Lifting abelian schemes: theorems of Serre-Tate and Grothendieck

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Remark 0.1. J'ai décidé d'écrire ces notes en anglais, car il y a trop d'accents auxquels il faut faire gaffe....

1 The Serre-Tate theorem

Let $S_0 \to S$ be a nilpotent thickening of schemes (so $S_0 \to S$ is a closed immersion whose ideal I is locally nilpotent). Suppose that p is a prime which is locally nilpotent on S. If X is an object over S, let X_0 be its base-change to S_0. If A/S is an abelian scheme, let $A[p^\infty]$ be its p-divisible group.

Theorem 1.1. *(Serre-Tate)*

a) Let A, B be abelian schemes over S and let $f_0 \in \mathrm{Hom}_{S_0}(A_0, B_0)$. Then f_0 has a lifting to some $f \in \mathrm{Hom}_S(A, B)$ if and only if $f_0[p^\infty] \in \mathrm{Hom}(A_0[p^\infty], B_0[p^\infty])$ has a lifting to some $f_\infty \in \mathrm{Hom}(A[p^\infty], B[p^\infty])$. If such liftings exist, then they are unique.

b) Let A_0/S_0 be an abelian scheme, let G/S be a p-divisible group and finally let $i : A_0[p^\infty] \to G_0$ be an isomorphism of p-divisible groups. Then one can find an abelian scheme A/S and an isomorphism $i : A[p^\infty] \to G$ such that (A, i) lifts (A_0, i_0). Moreover, the pair (A, i) is unique up to unique isomorphism.

The proof will span over the next subsections.

1.2 Formal completion of fppf abelian sheaves

A k-thickening of S is a closed immersion $X \to S$ whose defining ideal I satisfies $I^{k+1} = 0$.

Let G be an abelian fppf sheaf on a scheme S and let $k \geq 1$. Define the subsheaf $\mathrm{inf}^k(G)$ of G, whose sections over T (an S scheme) are those sections $t \in G(T)$ for which there is an fppf covering $\{T_i \to T\}$ and k-thickenings $X_i \to T_i$ such that $t = 0$ on X_i.

Example 1.3. Let G/S be a commutative group scheme with augmentation ideal I. Descent theory allows us to see G as an abelian fppf sheaf. Then one checks (first page of chapter II of Messing's book) that $\mathrm{inf}^k(G) = \mathrm{Spec}(O_G/I^{k+1})$.

Definition 1.4. The formal completion of G is the subsheaf of G defined by $\hat{G} = \lim_{\to} \mathrm{inf}^k(G)$.

Example 1.5. Let A be an abelian scheme over S. Then $\hat{A}$ is really the formal completion of A along the unit section of A and this is a formal Lie group by the usual theory of smooth group schemes (A being smooth, its completed local ring at the identity section is a formal power series ring and inherits operations via the operations on A).

Theorem 1.6. *(Grothendieck-Messing) Suppose that p is locally nilpotent on S and that G is a p-divisible group. Then G is formally smooth and $\hat{G}$ is a formal Lie group.*

Proof. This deep theorem is one of the main results of chapter II in Messing's book. □

Remark 1.7. 1) One has $G = \hat{G}$ if and only if $G[p]$ is radicial. In this case we also say that the p-divisible group G is formal.

2) Suppose that R is a complete local noetherian ring and that A is an abelian scheme over R. Then one can easily check that

$$\hat{A}[p^n] = A[p^n]^0, \quad \hat{A} = A[p^\infty]^0.$$

Definition 1.8. An abelian fppf sheaf G on S is called almost formal if $p : G \to G$ is an epimorphism and $\hat{G}$ is a formal Lie group.

So, if p is locally nilpotent on S, then p-divisible groups and abelian schemes are almost formal.

1.9 Drinfeld's rigidity lemma

From now on we assume, without loss of generality, that S (hence also S_0) is affine, say $S = \mathrm{Spec}(R)$ and $S_0 = \mathrm{Spec}(R_0)$. Let I be the kernel of the surjection $R \to R_0$ and assume that $I^{N+1} = 0$. Also, let $k \geq 1$ be such that $p^k R = 0$.

Lemma 1.10. *(Drinfeld) Let G be an almost formal abelian fppf sheaf on S. Then p^{nk} kills the kernel of the map $G(A) \to G(A/IA)$ for any R-algebra A.*

Proof. By definition of the formal completion, any element in the kernel lies in $\hat{G}(A)$. It is thus sufficient to prove that p^{nk} kills the kernel of $\hat{G}(A) \to \hat{G}(A/IA)$. Pick coordinates $X_1, ..., X_d$ on the formal Lie group $\hat{G}$ and let x in the kernel have coordinates $x_1, ..., x_d$. Then $x_i \in IA$ for all i, so that (since $p^k x_i = 0$) necessarily the coordinates of $p^k \cdot x$ live in I^2. Replacing I by any of its powers, we see that p^k sends the kernel of $\hat{G}(A) \to \hat{G}(A/I^j A)$ into the kernel of $\hat{G}(A) \to \hat{G}(A/I^{j+1}A)$ for all j. Applying this N times and using $I^{N+1} = 0$ yields the result.

□

Corollary 1.11. *(Drinfeld's rigidity lemma) Let G, H be almost formal abelian fppf sheaves on S. The natural map $\mathrm{Hom}_S(G, H) \to \mathrm{Hom}_{S_0}(G_0, H_0)$ is injective.*

Proof. Since multiplication by p is an epimorphism on an almost formal group, it suffices to prove that $p^{Nk} f = 0$ for any $f \in \mathrm{Hom}_S(G, H)$ which dies in $\mathrm{Hom}_{S_0}(G_0, H_0)$. But for any R-algebra A, the induced morphism $f : G(A) \to H(A)$ takes values in $\mathrm{Ker}(H(A) \to H(A/I))$, which is killed by p^{Nk} (previous lemma). Hence p^{Nk} kills f. □

1.12 "Canonical lifting"

Keep the same assumptions as in the previous section.

Proposition 1.13. *("canonical lifting") Let G, H be almost formal groups over R and let $f_0 \in \mathrm{Hom}_{S_0}(G_0, H_0)$. There exists a unique $f_N \in \mathrm{Hom}_S(G, H)$ lifting $p^{Nk} f_0$. Moreover, f_N kills $G[p^{Nk}]$ if and only if f_0 lifts to a homomorphism between G and H*

Proof. Define $f_N : G(A) \to H(A)$ (for an R-algebra A) as the composite

$$G(A) \to G(A/IA) \to H(A/IA) \to H(A),$$

where the middle map is induced by f_0 and the map on the right is $p^{Nk} s$, where s is any (set-theoretical) section of the surjective map $H(A) \to H(A/IA)$ (this map is onto since H is formally smooth). The resulting map f_N does not depend on the choice of s by lemma 1.10. By construction, the various f_N defined for various A yield an fppf homomorphism lifting $p^{Nk} f_0$.

Let us prove the second part. First, assume that f_0 lifts to some $f \in \mathrm{Hom}(G, H)$. Then $f_N = p^{Nk} f$, by Drinfeld's rigidity lemma and since both of them lift $p^{Nk} f_0$. Hence f_N kills $G[p^{Nk}]$. Conversely, suppose that f_N kills $G[p^{Nk}]$. Then we can find $f \in \mathrm{Hom}(G, H)$ such that $f_N = p^{Nk} f$ (indeed, multiplication by p^{Nk} induces an exact sequence of abelian fppf sheaves $0 \to G[p^{Nk}] \to G \to G \to 0$, which allows us to define f as $G \simeq G/G[p^{Nk}] \to H$). It remains to check that f lifts f_0. But p^{Nk} kills $f_0 - (f \pmod I)$ and it is immediate to see (use the same exact sequence) that this forces $f_0 = (f \pmod I)$. □

1.14 Proof of Serre-Tate's theorem

Let us prove 1) first. Drinfeld's rigidity lemma yields uniqueness of the liftings, in case they exist. One direction being clear, assume that $f_0[p^\infty]$ lifts to a morphism $g_\infty \in \mathrm{Hom}(A[p^\infty], B[p^\infty])$. Let $f_N \in \mathrm{Hom}(A, B)$ be the canonical lifting of $p^{Nk} f_0$. Then, by rigidity, we must have $f_N[p^\infty] = p^{Nk} g_\infty$ (both of them lift $p^{Nk} f_0[p^\infty]$. Hence f_N kills $A[p^{Nk}]$ and so f_0 lifts to some $f \in \mathrm{Hom}(A, B)$, finishing the proof.

Let us prove 2) now. See the next section for a proof (in a special case, which is however enough for most applications, including what we need for local Langlands) of the following wonderful:

Theorem 1.15. *(Grothendieck) If A_0/S_0 is an abelian scheme, then there exists an abelian scheme A/S whose base-change to S_0 is A_0.*

So, take an abelian scheme X/S such that $X_0 \simeq A_0$. This yields an isomorphism (which abusively will also be called) $i_0 : (X[p^\infty])_0 \simeq G_0$. Let $i_N : X[p^\infty] \to G$ be the canonical lifting of $p^{Nk} i_0$ and let $j_N : G \to X[p^\infty]$ be the canonical lifting of $p^{Nk} i_0^{-1}$. Then, by rigidity, we must have $i_N \circ j_N = p^{2Nk}$ and $j_N \circ i_N = p^{2Nk}$. Hence i_N is an isogeny and its kernel K is a finite subgroup of $X[p^{2Nk}]$. We claim that K is flat over R (hence a finite locally free subgroup of X). By base-change, it suffices to prove that i_N

is flat. This follows from the fibre-by-fibre criterion of flatness, the fact that $S_0 \to S$ has nilpotent ideal of definition and the fact that the base-change to S_0 of i_N is $p^{Nk} i_0$ (hence flat). This proves the claim. Now, standard results about quotients of group schemes (SGA3) show that $A = X/K$ is an abelian scheme over S. By contruction, it satisfies all desired properties. Uniqueness follows from rigidity.

2 Grothendieck's theorem on abelian schemes

I will make some extra-assumptions, which will be sufficient for our purposes. Actually, I think that with very mild extra-work we can get rid of them, but I'm lazy to do that...

Hence, we assume in this section that R is a local artinian ring with maximal ideal m and residue field $k = R/m$. Let $R_0 = R/I$, where I is an ideal killed by m (in particular $I^2 = 0$). It is easy to see that if R is artin local, then any surjection $R \to R_0$ (for some other artin local ring R_0) is a composite of finitely many extensions of the previous form (also called small extensions, following Schlessinger). The purpose of this section is to prove the following beautiful theorem:

Theorem 2.1. *(Grothendieck) Under the previous assumptions, any abelian scheme over R_0 lifts to an abelian scheme over R.*

Here are the main steps:

1) Deformation theory of smooth schemes combined with basic facts about cohomology of abelian schemes (most importantly the fact that the relative tangent sheaf is trivial) will show that any abelian scheme A_0/R_0 lifts to a smooth scheme A/R. Moreover, the identity section of A_0 lifts to a section of A.

2) One proves that ANY such smooth scheme A/R endowed with a section lifting the identity section of A_0 must be an abelian scheme. This is actually the nicest part of the proof.

All this needs some preliminaries...

2.2 Obstructions to lifting smooth schemes

Here we let $S_0 \to S$ be any closed immersion of schemes, whose ideal I satisfies $I^2 = 0$ (of course, we will then specialize to $S_0 = \mathrm{Spec}(R_0)$, etc). Let X_0/S_0 be a smooth scheme. A deformation of X_0 to S will be a pair consisting of a flat scheme X/S and an isomorphism $i : X_{S_0} \to X_0$ (subscript S_0 means base change). There is an obvious notion of isomorphism of deformations. Note that any such deformation X/S has to be smooth (exercise: use fibre criterion). Let

$$T_{X_0/S_0} = \mathrm{Hom}(\Omega^1_{X_0/S_0}, O_{X_0})$$

be the relative tangent sheaf of X_0. Since X_0/S_0 is smooth, $\Omega^1_{X_0/S_0}$ is a locally free O_{X_0}-module, hence so is T_{X_0/S_0}. We will see I as an O_{X_0}-module by pullback via the structure morphism $X_0 \to S_0$.

Theorem 2.3. *a) There is an obstruction $o(X_0) \in H^2(X_0, I \otimes T_{X_0/S_0})$, which vanishes if and only if there is at least one deformation of X_0 to S.*

b) If $o(X_0) = 0$, then the set of isomorphism classes of deformations of X_0 to S is a principal homogenous space under $H^1(X_0, I \otimes T_{X_0/S_0})$.

This theorem is actually "easy", but the proof is rather painful to write down. The reader should see Illusie's lecture notes on FGA, or chapter IV in SGA1 (suitably adapted to this more general setup).

Corollary 2.4. *If X_0/S_0 is etale, then there is a unique (up to unique isomorphism) deformation of X_0 to S and this deformation is etale over S.*

Proof. In this case $T_{X_0/S_0} = 0$, so everything is clear. □

The previous theorem will not be enough to show that abelian schemes over S_0 lift to smooth schemes over S. A key role will be played by the functorial properties of the obstruction:

Proposition 2.5. *1) Suppose that $f_0 : X_0 \to Y_0$ is an S_0-morphism of smooth S_0-schemes. Then f_0 induces two natural maps $H^2(X_0, I \otimes T_{X_0/S_0}) \to H^2(X_0, I \otimes f_0^* T_{Y_0/S_0})$ and $H^2(Y_0, I \otimes T_{Y_0/S_0}) \to H^2(X_0, I \otimes f_0^* T_{Y_0/S_0})$. The images of $o(X_0)$ and $o(Y_0)$ in $H^2(X_0, I \otimes f_0^* T_{Y_0/S_0})$ (via these maps) are the same.*

2) Let Y_0 be a smooth S_0 scheme and let pr_1 be the natural projection $X_0 \times_{S_0} Y_0 \to X_0$. Then pr_1 induces a map

$$H^2(X_0, I \otimes T_{X_0/S_0}) \to H^2(X_0 \times_{S_0} Y_0, I \otimes \mathrm{pr}_1^* T_{X_0/S_0}) \to H^2(X_0 \times_{S_0} Y_0, I \otimes T_{X_0 \times_{S_0} Y_0/S_0})$$

and we have an equality

$$o(X_0 \times_{S_0} Y_0) = \mathrm{pr}_1^*(o(X_0)) + \mathrm{pr}_2^*(o(Y_0)).$$

Proof. Easy, once you understood the proof of the previous theorem... □

2.6 Lifting abelian schemes to smooth schemes

Keep the initial assumptions (so R_0 is artin local, I is killed by m).

Proposition 2.7. *Let A_0 be an abelian scheme over R_0. Then A_0 lifts to a smooth scheme A over R.*

Proof. We need to prove that $o(A_0) = 0$. Let $B_0 = A_0 \times_{S_0} A_0$, an abelian scheme over $S_0 = \mathrm{Spec}(R_0)$. The two projections $B_0 \to A_0$ induce maps $\mathrm{pr}_j^* : H^2(A_0, I \otimes T_{A_0/S_0}) \to H^2(B_0, I \otimes T_{B_0/S_0})$ and $o(B_0) = \mathrm{pr}_1^*(o(A_0)) + \mathrm{pr}_2^*(o(A_0))$. A first key point is that pr_j^* are injective. This follows from the fact that the tangent sheaf of an abelian scheme is trivial (ie $T_{A_0/S_0} = O_{A_0} \otimes H^0(A_0, T_{A_0/S_0})$, from Kunneth's formula and the fact that $\mathrm{Lie}_{B_0/S_0} = \mathrm{Lie}_{A_0/S_0} \oplus \mathrm{Lie}_{A_0/S_0}$. This allows us to see $H^2(A_0, I \otimes T_{A_0/S_0})$ as a direct factor of $H^2(B_0, I \otimes T_{B_0/S_0})$ and the inclusions of this direct factor turn out to be the maps pr_j^* (by abstract nonsense...).

Next, consider the automorphism $\alpha(x,y) = (x+y,y)$ of B_0/S_0. This induces an automorphism α^* of $H^2(B_0, I \otimes T_{B_0/S_0})$ and by functoriality of obstructions we can write

$$o(B_0) = \alpha^*(o(B_0)) = \alpha^*(\mathrm{pr}_1^*(o(A_0)) + \mathrm{pr}_2^*(o(A_0))) = \mathrm{pr}_1^*(o(A_0)) + 2\mathrm{pr}_2^*(o(A_0)).$$

Hence $\mathrm{pr}_2^*(o(A_0)) = 0$. The previous paragraph shows that $o(A_0) = 0$, hence we are done.

□

2.8 Lifting the identity section of A_0

Since A/S is smooth, the identity section e_0 of A_0/S_0 lifts to a section e of A/S. Also, note that A/S is proper.

2.9 Any such A is an abelian scheme

Proposition 2.10. *Let A/S be a proper deformation of A_0, having a section e which lifts e_0. Then A/S is an abelian scheme.*

Proof. Let $B_0 = A_0 \times_{S_0} A_0$ and let $d_0 : B_0 \to A_0$ be the morphism $(x,y) \to x - y$. We will prove that d_0 lifts uniquely to an S-morphism d such that $d(e,e) = e$. This will endow A/S with addition and inversion and we will finally check that these operations satisfy the necessary compatibilities to ensure that A/S is an abelian scheme.

• With the same arguments as those used to lift schemes, one checks that the obstruction to lifting d_0 to a morphism of S-schemes $B \to A$ (here $B = A \times_S A$) lives in $H^1(B_0, I \otimes d_0^* T_{A_0/S_0})$. Note that I is a finite dimensional k-vector space and that d_0 induces the identity on the first factor A_0 of B_0. Hence

$$H^1(B_0, I \otimes d_0^* T_{A_0/S_0}) \simeq H^1(B_0, O_{B_0}) \otimes_k \mathrm{Lie}_{A_0/S_0} \otimes_k I \simeq$$

$$(\mathrm{pr}_1^* H^1(A_0, O_{A_0}) \otimes \mathrm{Lie}_{A_0/S_0} \otimes I) \oplus (\mathrm{pr}_2^* H^1(A_0, O_{A_0}) \otimes \mathrm{Lie}_{A_0/S_0} \otimes I),$$

the last isomorphism coming from Kunneth's formula. We will prove that the projections of $o(d_0)$ on these two direct factors are zero, which will show that $o(d_0) = 0$ and so that d_0 lifts. Let $i_1 : A_0 \to B_0$ be the map $x \to (x, e_0)$ and let $i_2 : A_0 \to B_0$ be the map $x \to (x,x)$. Note that both maps $d_0 \circ i_1$ and $d_0 \circ i_2$ lift to A (simply take the identity, respectively e). Hence $i_1^*(o(d_0)) = 0$ and $i_2^*(o(d_0)) = 0$. But since $\mathrm{pr}_1 \circ i_1 = \mathrm{id}$ and $\mathrm{pr}_2 \circ i_2 = \mathrm{id}$, it is easy to see that this implies $\mathrm{pr}_1^*(o(d_0)) = 0$ and $\mathrm{pr}_2^*(o(d_0)) = 0$, hence $o(d_0) = 0$.

• We claim that there is a unique lifting $d : B \to A$ of d_0 such that $d(e,e) = e$. The set of isomorphism classes of liftings of d_0 is a principal homogeneous space under $H^0(B_0, d_0^* T_{A_0/S_0}) \otimes_k I \simeq \mathrm{Lie}_{A_0/S_0} \otimes_k I$. The restrictions of the liftings of d_0 to (e,e) form a principal homogeneous space under $H^0(S_0, (d_0|_{(e_0,e_0)})^* T_{A_0/S_0}) \otimes_k I \simeq \mathrm{Lie}_{A_0/S_0} \otimes_k I$, hence the result.

• This special lift d of d_0 allows us to define addition and inverse on A, via the usual formulae. It remains to check that A/S becomes an abelian scheme when endowed with these operations. This comes down to checking a whole series of identities, all of which have the following form: we are given a morphism of S-schemes $f : A \times_S A \times_S \times ... \times_S A \to A$ such that $f(e, e, ..., e) = e$ and which is trivial modulo I (since we know that A_0/S_0 is an abelian scheme). We need to prove that f is trivial. The following rigidity lemma of Mumford shows that one can find a section $s : S \to A$ such that f is the composite $A \times_S A \times_S A \times ... \times_S A \to S \to A$ (the first map being the structural morphism). The condition $f(e, e, ..., e) = e$ implies that $s = e$ and the result follows.

Theorem 2.11. *(Mumford's rigidity lemma) Suppose that $f : X \to Y$ is a morphism of S schemes with S connected, $\pi : X \to S$ being flat, closed and such that $O_S \simeq \pi_* O_X$. Suppose that $\pi : X \to S$ has a section and that there is $s \in S$ such that $f(X_s)$ has one element. Then one can find a section s of the structural morphism $Y \to S$ such that $f = s \circ \pi$.*

□